Differential Geometry of Microlinear Frölicher Spaces III

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Abstract

As the third of our series of papers on differential geometry of microlinear Frölicher spaces is this paper devoted to the Frölicher-Nijenhuis calculus of their named bracket. The main result is that the Frölicher-Nijenhuis bracket satisfies the graded Jacobi identity. It is also shown that the Lie derivation preserves the Frölicher-Nijenhuis bracket. Our definitions and discussions are highly geometric, while Frölicher and Nijenhuis' original definitions and discussions were highly algebraic.

1 Introduction

As Mangiarotti and Modugno [10] have amply demonstrated, the central part of orthodox differential geometry based on principal connections can be developed within a more general framework of fibered manifolds (without any distinguished additional structures), in which the graded Lie algebra of tangent-vector-valued forms investigated by Frölicher and Nijenhuis [1] renders an appropriate differential calculus. The present paper is concerned with this graded Lie algebra, which plays a crucial role in their general differential geometry. Our present approach as well as [15] is highly combinatorial or geometric, while Frölicher and Nijenhuis' original approach was tremendously algebraic.

This paper consists of 4 sections, besides this introduction. The second section is devoted to preliminaries including vector fields and (real-valued) differential forms. Obviously tangent-vector-valued forms are a generalization of differential forms and vector fields at the same time, while the graded Lie algebra of tangent-vector-valued forms is a generaliztion of the Lie algebra of vector fields. Section 3 gives two distinct but equivalent views of tangent-vector-valued forms, just as we have given two distinct but equivalent views of vector fields in [18]. Section 4 is divided into two subsections, the first of which is mainly concerned with the graded Jacobi identity of entities much more general than

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tangent-vector-valued forms (i.e., without homogeneity or the alternating property assumed at all), while the second of which derives the graded Jacobi identity of tangent-vector-valued forms from the highly general graded Jacobi identity established in the first subsection. Our proof of the graded Jacobi identity in the first subsection is based upon the general Jacobi identity established by the author [13] more than a decade ago. Section 5 is similarly divided into two subsections, the first of which shows that the Lie derivation of differential semiforms by tangent-vector-valued semiforms preserves the Lie bracket, while the second of which demonstrates that the Lie derivation of differential forms by tangent-vector-valued forms preserves the Frölicher-Nijenhuis bracket.

2 Preliminaries

In this paper, $n, p, q, ...$ represent natural numbers. We assume that the reader has already read our previous papers [18] and [19].

2.1 Frölicher Spaces

Frölicher and his followers have vigorously and consistently developed a general theory of smooth spaces, often called *Frölicher spaces* for his celebrity, which were intended to be the *underlying set theory* for infinite-dimensional differential geometry. A Frölicher space is an underlying set endowed with a class of real-valued functions on it (simply called *structure functions*) and a class of mappings from the set $\mathbb{R}$ of real numbers to the underlying set (called *structure curves*) subject to the condition that structure curves and structure functions should compose so as to yield smooth mappings from $\mathbb{R}$ to itself. It is required that the class of structure functions and that of structure curves should determine each other so that each of the two classes is maximal with respect to the other as far as they abide by the above condition. What is most important among many nice properties about the category **FS** of Frölicher spaces and smooth mappings is that it is cartesian closed, while neither the category of finite-dimensional smooth manifolds nor that of infinite-dimensional smooth manifolds modelled after any infinite-dimensional vector spaces such as Hilbert spaces, Banach spaces, Fréchet spaces or the like is so at all. For a standard reference on Frölicher spaces the reader is referred to [5].

2.2 Weil Algebras and Infinitesimal Objects

The notion of a *Weil algebra* was introduced by Weil himself in [22]. We denote by **W** the category of Weil algebras. Roughly speaking, each Weil algebra corresponds to an infinitesimal object in the shade. By way of example, the Weil algebra $\mathbb{R}[X]/(X^2)$ (=the quotient ring of the polynomial ring $\mathbb{R}[X]$ of an indeterminate X over $\mathbb{R}$ modulo the ideal (X^2) generated by X^2) corresponds to the infinitesimal object of first-order nilpotent infinitesimals, while the Weil algebra

$\mathbb{R}[X]/(X^3)$ corresponds to the infinitesimal object of second-order nilpotent infinitesimals. Although an infinitesimal object is undoubtedly imaginary in the real world, as has harassed both mathematicians and philosophers of the 17th and the 18th centuries (because mathematicians at that time preferred to talk infinitesimal objects as if they were real entities), each Weil algebra yields its corresponding *Weil functor* on the category of smooth manifolds of some kind to itself, which is no doubt a real entity. By way of example, the Weil algebra $\mathbb{R}[X]/(X^2)$ yields the tangent bundle functor as its corresponding Weil functor. Intuitively speaking, the Weil functor corresponding to a Weil algebra stands for the exponentiation by the infinitesimal object corresponding to the Weil algebra at issue. For Weil functors on the category of finite-dimensional smooth manifolds, the reader is referred to §35 of [7], while the reader can find a readable treatment of Weil functors on the category of smooth manifolds modelled on convenient vector spaces in §31 of [8].

Synthetic differential geometry (usually abbreviated to SDG), which is a kind of differential geometry with a cornucopia of nilpotent infinitesimals, was forced to invent its models, in which nilpotent infinitesimals were visible. For a standard textbook on SDG, the reader is referred to [9], while he or she is referred to [6] for the model theory of SDG constructed vigorously by Dubuc [**?**] and others. Although we do not get involved in SDG herein, we will exploit locutions in terms of infinitesimal objects so as to make the paper highly readable. Thus we prefer to write $\mathcal{W}_D$ and $\mathcal{W}_{D_2}$ in place of $\mathbb{R}[X]/(X^2)$ and $\mathbb{R}[X]/(X^3)$ respectively, where D stands for the infinitesimal object of first-order nilpotent infinitesimals, and D_2 stands for the infinitesimal object of second-order nilpotent infinitesimals. To Newton and Leibniz, D stood for

$$\{d \in \mathbb{R} \mid d^2 = 0\}$$

while D_2 stood for

$$\{d \in \mathbb{R} \mid d^3 = 0\}$$

We will write $\mathcal{W}_{d \in D_2 \mapsto d^2 \in D}$ for the homomorphim of Weil algebras $\mathbb{R}[X]/(X^2) \rightarrow \mathbb{R}[X]/(X^3)$ induced by the homomorphism $X \rightarrow X^2$ of the polynomial ring $\mathbb{R}[X]$ to itself. Such locutions are justifiable, because the category $\mathbf{W}$ of Weil algebras in the real world and the category of infinitesimal objects in the shade are dual to each other in a sense. Thus we have a contravariant functor $\mathcal{W}$ from the category of infinitesimal objects in the shade to the category of Weil algebras in the real world. Its inverse contravariant functor from the category of Weil algebras in the real world to the category of Weil algebras in the real world is denoted by $\mathcal{D}$. By way of example, $\mathcal{D}_{\mathbb{R}[X]/(X^2)}$ and $\mathcal{D}_{\mathbb{R}[X]/(X^3)}$ stand for D and D_2 respectively. To familiarize himself or herself with such locutions, the reader is strongly encouraged to read the first two chapters of [9], even if he or she is not interested in SDG at all.

In [16] we have discussed how to assign, to each pair (X, W) of a Frölicher space X and a Weil algebra W, another Frölicher space $X \otimes W$ called the *Weil prolongation of X with respect to W*, which is naturally extended to a bifunctor $\mathbf{FS} \times \mathbf{W} \rightarrow \mathbf{FS}$, and then to show that the functor $\cdot \otimes W : \mathbf{FS} \rightarrow \mathbf{FS}$

is product-preserving for any Weil algebra W. Weil prolongations are well-known as *Weil functors* for finite-dimensional and infinite-dimensional smooth manifolds in orthodox differential geometry, as we have already discussed above.

The central object of study in SDG is *microlinear* spaces. Although the notion of a manifold (=a pasting of copies of a certain linear space) is defined on the local level, the notion of microlinearity is defined on the genuinely infinitesimal level. For the historical account of microlinearity, the reader is referred to §§2.4 of [9] or Appendix D of [6]. To get an adequately restricted cartesian closed subcategory of Frölicher spaces, we have emancipated microlinearity from within a well-adapted model of SDG to Frölicher spaces in the real world in [17]. Recall that a Frölicher space X is called *microlinear* providing that any finite limit diagram $\mathbb{D}$ in $\mathbf{W}$ yields a limit diagram $X \otimes \mathbb{D}$ in $\mathbf{FS}$, where $X \otimes \mathbb{D}$ is obtained from $\mathbb{D}$ by putting $X \otimes$ to the left of every object and every morphism in $\mathbb{D}$.

As we have discussed there, all convenient vector spaces are microlinear, so that all C^∞-manifolds in the sense of [8] (cf. Section 27) are also microlinear.

We have no reason to hold that all Frölicher spaces credit Weil prolongations as exponentiation by infinitesimal objects in the shade. Therefore we need a notion which distinguishes Frölicher spaces that do so from those that do not. A Frölicher space X is called *Weil exponentiable* if

$$(X \otimes (W_1 \otimes_\infty W_2))^Y = (X \otimes W_1)^Y \otimes W_2 \tag{1}$$

holds naturally for any Frölicher space Y and any Weil algebras W_1 and W_2. If $Y = 1$, then (1) degenerates into

$$X \otimes (W_1 \otimes_\infty W_2) = (X \otimes W_1) \otimes W_2 \tag{2}$$

If $W_1 = \mathbb{R}$, then (1) degenerates into

$$(X \otimes W_2)^Y = X^Y \otimes W_2 \tag{3}$$

We have shown in [16] that all convenient vector spaces are Weil exponentiable, so that all C^∞-manifolds in the sense of [8] (cf. Section 27) are Weil exponentiable.

We have demonstrated in [17] that all Frölicher spaces that are microlinear and Weil exponentiable form a cartesian closed category. In the sequel, M shall be assumed to be such a Frölicher space.

It is well known that the category $\mathbf{W}$ is left exact. In SDG, a finite diagram $\mathbb{D}$ in $\mathbf{D}$ is called a *quasi-colimit diagram* provided that the contravariant functor $\mathcal{W}$ transforms $\mathbb{D}$ into a limit diagram in $\mathbf{W}$. By way of example, the following diagram in $\mathbf{D}$ is a famous quasi-colimit diagram, for which the reader is referred to pp.92-93 of [9].

$$
\begin{array}{ccc}
& i & \\
D(2) & \rightarrow & D^2 \\
\\
i \downarrow & & \downarrow \psi \qquad\qquad (4)\\
\\
D^2 & \rightarrow & D^2 \oplus D \\
& \varphi &
\end{array}
$$

where $i : D(2) \to D^2$ is the canonical injection, $D^2 \oplus D$ is

$$
D^2 \oplus D = \{(d_1, d_2, e) \in D^3 \mid d_1 e = d_2 e = 0\}
$$

$\varphi : D^2 \to D^2 \oplus D$ is

$$
\varphi(d_1, d_2) = (d_1, d_2, 0)
$$

for any $(d_1, d_2) \in D^2$, and $\psi : D^2 \to D^2 \oplus D$ is

$$
\psi(d_1, d_2) = (d_1, d_2, d_1 d_2)
$$

for any $(d_1, d_2) \in D^2$.

2.3   Vector Fields

Our two distinct but equivalent viewpoints of vector fields on M are simply based upon the following exponential law:

$$
\begin{aligned}
&[M \to M \otimes \mathcal{W}_D] \\
&= [M \to M] \otimes \mathcal{W}_D
\end{aligned}
$$

The first definition of a vector field on M goes as follows:

Definition 1 *A vector field on M is a section of the tangent bundle $\pi : M \otimes \mathcal{W}_D \to M$.*

The second definition of a vector field on M goes as follows:

Definition 2 *A vector field on M is a tangent vector of the space $[M \to M]$ with foot point id_M.*

Generally speaking, we prefer the second definition of a vector field to the first one. In our previous paper [18], we have shown that

Theorem 3 *The totality of vector fields on M forms a Lie algebra.*

In particular, our proof of the Jacobi identity of vector fields is based upon the following general Jacobi identity.

Theorem 4 *Let* $\gamma_{123}, \gamma_{132}, \gamma_{213}, \gamma_{231}, \gamma_{312}, \gamma_{321} \in M \otimes W_{D^3}$. *As long as the following three expressions are well defined, they sum up only to vanish:*

$$(\gamma_{123} \underset{1}{\overset{.}{-}} \gamma_{132}) - (\gamma_{231} \underset{1}{\overset{.}{-}} \gamma_{321})$$

$$(\gamma_{231} \underset{2}{\overset{.}{-}} \gamma_{213}) - (\gamma_{312} \underset{2}{\overset{.}{-}} \gamma_{132})$$

$$(\gamma_{312} \underset{3}{\overset{.}{-}} \gamma_{321}) - (\gamma_{123} \underset{3}{\overset{.}{-}} \gamma_{213})$$

The above theorem was discovered by the author in [13] more than a decade ago, and with due regard to its importance, it was provided with two other proofs in [14] and [21].

2.4 Euclidean Vector Spaces

Frankly speaking, our exposition of a Euclidean vector space in [19] was a bit confused. The exact definition of a Euclidean vector space goes as follows.

Definition 5 *A vector space* $\mathbb{E}$ *(over* $\mathbb{R}$*) in the category* **FS** *is called Euclidean provided that the canonical mapping* $\mathbf{i}_{\mathbb{E}}^1 : \mathbb{E} \times \mathbb{E} \to \mathbb{E} \otimes \mathcal{W}_D$ *induced by the mapping*

$$(\mathbf{a}, \mathbf{b}) \in \mathbb{E} \times \mathbb{E} \mapsto (x \in \mathbb{R} \mapsto \mathbf{a} + x\mathbf{b} \in \mathbb{E}) \in \mathbb{E}^{\mathbb{R}}$$

is bijective.

Notation 6 *Let* $\mathbb{E}$ *be a Euclidean vector space. Given* $\gamma \in \mathbb{E} \otimes \mathcal{W}_D$*, we write* $\mathbf{D}(\gamma)$ *for* $\mathbf{b} \in \mathbb{E}$ *in the above definition.*

Notation 7 *Let* $\mathbb{E}$ *be a Euclidean vector space. Given* $\gamma \in \mathbb{E} \otimes \mathcal{W}_{D^n}$ *(* $n \geq 2$*), we write* $\mathbf{D}_i(\gamma) \in \mathbb{E} \otimes \mathcal{W}_{D^{n-1}}$ *for the image of* γ *under the composite of mappings*

$$\mathbb{E} \otimes \mathcal{W}_{D^n} \xrightarrow{\mathrm{id}_{\mathbb{E}} \otimes \mathcal{W}_{(d_1, \ldots d_n) \in D^n \mapsto (d_1, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n, d_i) \in D^n}} \mathbb{E} \otimes \mathcal{W}_{D^n}$$

$$= \mathbb{E} \otimes (\mathcal{W}_{D^{n-1}} \otimes_\infty \mathcal{W}_D) = (\mathbb{E} \otimes \mathcal{W}_{D^{n-1}}) \otimes \mathcal{W}_D \xrightarrow{\mathbf{D}} \mathbb{E} \otimes \mathcal{W}_{D^{n-1}}$$

Theorem 8 *In a Euclidean vector space* $\mathbb{E}$*, Taylor's expansion theorem holds in the sense that the canonical mapping* $\mathbf{i}_{\mathbb{E}}^2 : \mathbb{E} \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \to \mathbb{E} \otimes \mathcal{W}_{D^2}$ *induced by the mapping*

$$(\mathbf{a}, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_{12}) \in \mathbb{E} \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \mapsto$$

$$\left((x_1, x_2) \in \mathbb{R}^2 \mapsto \mathbf{a} + x_1 \mathbf{b}_1 + x_2 \mathbf{b}_2 + x_1 x_2 \mathbf{b}_{12} \in \mathbb{E} \right) \in \mathbb{E}^{\mathbb{R}^2}$$

is bijective, the canonical mapping $\mathbf{i}_{\mathbb{E}}^3 : \mathbb{E} \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \to \mathbb{E} \otimes \mathcal{W}_{D^3}$ *induced by the mapping*

$$(\mathbf{a}, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_{12}, \mathbf{b}_{13}, \mathbf{b}_{23}, \mathbf{b}_{123}) \in \mathbb{E} \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \mapsto$$

$$\left((x_1, x_2, x_3) \in \mathbb{R}^3 \mapsto \mathbf{a} + x_1 \mathbf{b}_1 + x_2 \mathbf{b}_2 + x_3 \mathbf{b}_3 \right.$$

$$\left. + x_1 x_2 \mathbf{b}_{12} + x_1 x_3 \mathbf{b}_{13} + x_2 x_3 \mathbf{b}_{23} + x_1 x_2 x_3 \mathbf{b}_{123} \in \mathbb{E} \right) \in \mathbb{E}^{\mathbb{R}^3}$$

is bijective, and so on.

Proof. Here we deal only with the first case, leaving similar treatments of the other cases to the reader. Schematically we have

$$\mathbb{E} \otimes \mathcal{W}_{D^2}$$
$$= \mathbb{E} \otimes (\mathcal{W}_D \otimes_\infty \mathcal{W}_D)$$
$$= (\mathbb{E} \otimes \mathcal{W}_D) \otimes \mathcal{W}_D$$
$$= (\mathbb{E} \times \mathbb{E}) \otimes \mathcal{W}_D$$
$$= (\mathbb{E} \otimes \mathcal{W}_D) \times (\mathbb{E} \otimes \mathcal{W}_D)$$

[Since the endofunctor $\cdot \otimes \mathcal{W}_D$ of the category **FS** preserves products]

$$= \mathbb{E} \times \mathbb{E} \times \mathbb{E} \times \mathbb{E}$$

∎

Proposition 9 *Let $\mathbb{E}$ be a Euclidean vector space. Given $\gamma \in \mathbb{E} \otimes \mathcal{W}_{D^2}$, we have*

$$\mathbf{D}\left(\mathbf{D}_2\left(\gamma\right)\right) = \mathbf{D}\left(\mathbf{D}_1\left(\gamma\right)\right)$$

Proof. It is easy to see that both sides give rise to $\mathbf{b}_{12}$ in Theorem 8. ∎
Here we will give a slight variant of Taylor's expansion theorem.

Theorem 10 *Let $\mathbb{E}$ be a Euclidean vector space, which is microlineaar. The canonical mapping $\mathbf{i}_{\mathbb{E}}^{D^2 \oplus D} : \mathbb{E} \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \to \mathbb{E} \otimes \mathcal{W}_{D^2 \oplus D}$ induced by the mapping*

$$(\mathbf{a}, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_{12}, \mathbf{c}) \in \mathbb{E} \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \mapsto$$

$$\left((x_1, x_2, x_3) \in \mathbb{R}^3 \mapsto \mathbf{a} + x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_1 x_2 \mathbf{b}_{12} + x_3\mathbf{c} \in \mathbb{E}\right) \in \mathbb{E}^{\mathbb{R}^3}$$

is bijective.

Proof. This follows from Theorem 8 and the quasi-colimit diagram (4). ∎

Proposition 11 *Let $\mathbb{E}$ be a Euclidean vector space, which is microlineaar. Let $\gamma_1, \gamma_2 \in \mathbb{E} \otimes \mathcal{W}_{D^2}$ with*

$$\left(\mathrm{id}_{\mathbb{E}} \otimes \mathcal{W}_{(d_1, d_2) \in D(2) \mapsto (d_1, d_2) \in D^2}\right)(\gamma_1)$$
$$= \left(\mathrm{id}_{\mathbb{E}} \otimes \mathcal{W}_{(d_1, d_2) \in D(2) \mapsto (d_1, d_2) \in D^2}\right)(\gamma_2)$$

Then we have

$$\mathbf{D}\left(\gamma_1 \overset{\cdot}{-} \gamma_2\right) = \mathbf{D}\left(\mathbf{D}_2\left(\gamma_1\right)\right) - \mathbf{D}\left(\mathbf{D}_2\left(\gamma_2\right)\right)$$

Proof. Let the Taylor's expansion of γ_1 be

$$(x_1, x_2) \in \mathbb{R}^2 \mapsto \mathbf{a} + x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + x_1 x_2 \mathbf{b}_{12} \in \mathbb{E}$$

with

$$(\mathbf{a}, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_{12}) \in \mathbb{E} \times \mathbb{E} \times \mathbb{E} \times \mathbb{E}$$

and the Taylor's expansion of γ_2 be

$$(x_1, x_2) \in \mathbb{R}^2 \mapsto \mathbf{a} + x_1 \mathbf{b_1} + x_2 \mathbf{b_2} + x_1 x_2 \mathbf{b}'_{12} \in \mathbb{E}$$

with $\mathbf{b}'_{12} \in \mathbb{E}$ and $\mathbf{a}, \mathbf{b}_1, \mathbf{b}_2$ being the same as above. Then the Taylor's expansion of $\gamma \in \mathbb{E} \otimes \mathcal{W}_{D^2 \oplus D}$ with $(\mathrm{id}_\mathbb{E} \otimes \mathcal{W}_\varphi)(\gamma) = \gamma_2$ and $(\mathrm{id}_\mathbb{E} \otimes \mathcal{W}_\psi)(\gamma) = \gamma_1$ is

$$(x_1, x_2, x_3) \in \mathbb{R}^3 \mapsto \mathbf{a} + x_1 \mathbf{b_1} + x_2 \mathbf{b_2} + x_1 x_2 \mathbf{b}_{12} + x_3 (\mathbf{b}_{12} - \mathbf{b}'_{12}) \in \mathbb{E}$$

so that

$$\mathbf{D}\left(\gamma_1 \dot{-} \gamma_2\right) = \mathbf{b}_{12} - \mathbf{b}'_{12}$$

which completes the proof. ∎

2.5 Differential Forms

We recall the familiar definition.

Definition 12 *An element θ of the space $[M \otimes \mathcal{W}_{D^n} \to \mathbb{R}]$ is called a (real-valued) differential n-form provided that*

    1. θ is n-homogeneous in the sense that

$$\theta\left(\alpha \underset{i}{\cdot} \theta\right) = \alpha\theta(\gamma)$$

    for any $\gamma \in M \otimes W_{D^n}$ and any $\alpha \in \mathbb{R}$, where $\alpha \underset{i}{\cdot} \gamma$ is defined by

$$\alpha \underset{i}{\cdot} \gamma = \left(\mathrm{id}_M \otimes \mathcal{W}_{\left(\alpha \underset{i}{\cdot}\right)_{D^n}}\right)(\gamma)$$

    with the putative mapping $\left(\alpha \underset{i}{\cdot}\right)_{D^n} : D^n \to D^n$ being

$$(d_1, ..., d_n) \in D^n \mapsto (d_1, ..., d_{i-1}, \alpha d_i, d_{i+1}, ..., d_n) \in D^n$$

    2. θ is alternating in the sense that

$$\omega(\gamma^\sigma) = \epsilon_\sigma \omega(\gamma)$$

    for any $\sigma \in \mathbb{S}_n$, where $\mathbb{S}_n$ is the group of permutations of $1, ..., n$, ϵ_σ is the sign of the permutation σ, and γ^σ is defined by

$$\gamma^\sigma = (\mathrm{id}_M \otimes \mathcal{W}_{\sigma_{D^n}})(\gamma)$$

    with the putative mapping $\sigma_{D^n} : D^n \to D^n$ being

$$(d_1, ..., d_n) \in D^n \mapsto (d_{\sigma(1)}, ..., d_{\sigma(n)}) \in D^n$$

Definition 13 *By dropping the second condition in the above definition, we get the notion of a differential n-semiform on M.*

Notation 14 *We denote by $\Omega^n(M)$ and $\widetilde{\Omega}^n(M)$ the totality of differential n-forms on M and that of differential n-semiforms on M, respectively. We denote by $\Omega(M)$ and $\widetilde{\Omega}(M)$ the totality of differential forms on M and that of differential semiforms on M, respectively.*

Definition 15 *Given $\theta_1 \in [M \otimes \mathcal{W}_{D^p} \to \mathbb{R}]$ and $\theta_2 \in [M \otimes \mathcal{W}_{D^q} \to \mathbb{R}]$, we define $\theta_1 \otimes \theta_2 \in [M \otimes \mathcal{W}_{D^{p+q}} \to \mathbb{R}]$ to be*

$$(\theta_1 \otimes \theta_2)(\gamma)$$
$$= \theta_1\left(\mathcal{W}_{(d_1,\ldots,d_p)\in D^p \mapsto (d_1,\ldots,d_p,0,\ldots,0)\in D^{p+q}}(\gamma)\right) \theta_2\left(\mathcal{W}_{(d_1,\ldots,d_q)\in D^q \mapsto (0,\ldots,0,d_1,\ldots,d_q)\in D^{p+q}}(\gamma)\right)$$

It is easy to see the following.

Proposition 16 *If θ_1 is a differential p-semiform on M and θ_2 is a differential q-semiform on M, then $\theta_1 \otimes \theta_2$ is a differential $(p+q)$-semiform on M.*

Proposition 17 *Given $\theta_1 \in [M \otimes \mathcal{W}_{D^p} \to \mathbb{R}]$, $\theta_2 \in [M \otimes \mathcal{W}_{D^q} \to \mathbb{R}]$ and $\theta_3 \in [M \otimes \mathcal{W}_{D^r} \to \mathbb{R}]$, we have*

$$(\theta_1 \otimes \theta_2) \otimes \theta_3 = \theta_1 \otimes (\theta_2 \otimes \theta_3)$$

Remark 18 *Therefore we can write $\theta_1 \otimes \theta_2 \otimes \theta_3$ without ambiguity.*

Definition 19 *Given $\theta \in [M \otimes \mathcal{W}_{D^p} \to \mathbb{R}]$, we define $\mathcal{A}\theta \in [M \otimes \mathcal{W}_{D^p} \to \mathbb{R}]$ to be*

$$\mathcal{A}\theta = \sum_{\sigma \in \mathbb{S}_p} \varepsilon_\sigma \theta^\sigma$$

Notation 20 *Given $\theta \in [M \otimes \mathcal{W}_{D^{p+q}} \to \mathbb{R}]$, we write $\mathcal{A}_{p,q}\theta$ for $(1/p!q!)\mathcal{A}\theta$. Given $\theta \in [M \otimes \mathcal{W}_{D^{p+q+r}} \to \mathbb{R}]$, we write $\mathcal{A}_{p,q,r}\theta$ for $(1/p!q!r!)\mathcal{A}\theta$.*

Definition 21 *Given $\theta_1 \in [M \otimes \mathcal{W}_{D^p} \to \mathbb{R}]$ and $\theta_2 \in [M \otimes \mathcal{W}_{D^q} \to \mathbb{R}]$, we define $\theta_1 \wedge \theta_2 \in [M \otimes \mathcal{W}_{D^{p+q}} \to \mathbb{R}]$ to be $\mathcal{A}_{p,q}(\theta_1 \otimes \theta_2)$.*

It is easy to see the following.

Proposition 22 *If θ is a differential semiform on M, then $\mathcal{A}\theta$ is a differential form.*

Proposition 23 *Given $\theta_1 \in [M \otimes \mathcal{W}_{D^p} \to \mathbb{R}]$, $\theta_2 \in [M \otimes \mathcal{W}_{D^q} \to \mathbb{R}]$ and $\theta_3 \in [M \otimes \mathcal{W}_{D^r} \to \mathbb{R}]$, we have*

$$\begin{aligned}
&\mathcal{A}_{p,q+r}(\theta_1 \otimes \mathcal{A}_{q,r}(\theta_2 \otimes \theta_3)) \\
&= \mathcal{A}_{p+q,r}(\mathcal{A}_{p,q}(\theta_1 \otimes \theta_2) \otimes \theta_3) \\
&= \mathcal{A}_{p,q,r}(\theta_1 \otimes \theta_2 \otimes \theta_3)
\end{aligned}$$

Corollary 24 *Given* $\theta_1 \in [M \otimes \mathcal{W}_{D^p} \to \mathbb{R}]$, $\theta_2 \in [M \otimes \mathcal{W}_{D^q} \to \mathbb{R}]$ *and* $\theta_3 \in [M \otimes \mathcal{W}_{D^r} \to \mathbb{R}]$, *we have*

$$(\theta_1 \wedge \theta_2) \wedge \theta_3 = \theta_1 \wedge (\theta_2 \wedge \theta_3)$$

It is easy to see the following two propositions.

Proposition 25 *Convenient vector spaces are Euclidean vector spaces which are microlinear and Weil exponentiable.*

Proposition 26 *The spaces* $\Omega^n(M)$ *and* $\widetilde{\Omega}^n(M)$ *are convenient vector spaces.*

Therefore we have

Proposition 27 *The spaces* $\Omega^n(M)$ *and* $\widetilde{\Omega}^n(M)$ *are Euclidean vector spaces which are microlinear.*

3 Tangent-Vector-Valued Differential Forms

Our two distinct but equivalent viewpoints of tangent-vector-valued differential forms on M are based upon the following exponential law:

$$[M \otimes \mathcal{W}_{D^p} \to M \otimes \mathcal{W}_D]$$
$$= [M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_D$$

If $p = 0$, the above law degenerates into the corresponding one in §**??**.

The first viewpoint, which is highly orthodox, goes as follows.

Definition 28 *A tangent-vector-valued p-form on* M *is a mapping* $\xi : M \otimes \mathcal{W}_{D^p} \to M \otimes \mathcal{W}_D$ *subject to the following three conditions:*

1. *We have*
$$\pi_M^{M \otimes \mathcal{W}_{D^p}}(\gamma) = \pi_M^{M \otimes \mathcal{W}_D}(\xi(\gamma))$$

   for any $\gamma \in M \otimes \mathcal{W}_{D^p}$.

2. *We have*
$$\xi(\alpha \underset{i}{\cdot} \gamma) = \alpha\xi(\gamma)$$

   for any $\alpha \in \mathbb{R}$, *any* $\gamma \in M \otimes \mathcal{W}_{D^p}$ *and any natural number* i *with* $1 \leq i \leq p$.

3. *We have*
$$\xi(\gamma^\sigma) = \varepsilon_\sigma\xi(\gamma)$$

   for any $\gamma \in M \otimes \mathcal{W}_{D^p}$ *and any* $\sigma \in \mathbb{S}_p$.

By dropping the third condition, we get the weaker notion of a tangent-vector-valued p-semiform on M.

The other viewpoint, which is highly radical, goes as follows.

Definition 29 *A tangent-vector-valued p-form on M is an element $\xi \in [M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_D$ pursuant to the following three conditions:*

1. *We have*
$$\pi^{[M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_D}_{[M \otimes \mathcal{W}_{D^p} \to M]}(\xi) = \delta^p_M$$
   where $\pi^{[M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_D}_{[M \otimes \mathcal{W}_{D^p} \to M]} : [M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_D \to [M \otimes \mathcal{W}_{D^p} \to M]$ is the canonical projection, and δ^p_M, called a (p-dimensional) Dirac distribution on M, denotes the canonical projection $\pi^{M \otimes \mathcal{W}_{D^p}}_M : M \otimes \mathcal{W}_{D^p} \to M$.

2. *We have*
$$\left(\left(\left(\alpha \underset{i}{\cdot} \right)_{M \otimes \mathcal{W}_{D^p}} \right)^* \otimes \mathrm{id}_{\mathcal{W}_D} \right)(\xi) = \alpha\xi$$

3. *We have*
$$\left(\left((\cdot^\sigma)_{M \otimes \mathcal{W}_{D^p}} \right)^* \otimes \mathrm{id}_{\mathcal{W}_D} \right)(\xi) = \varepsilon_\sigma \xi$$

By dropping the third condition, we get the weaker notion of a tangent-vector-valued p-semiform on M.

The following proposition is simple but very important and highly useful.

Proposition 30 *The addition for tangent-vector-valued p-semiforms on M in the first sense (i.e., using the fiberwise addition of the vector bundle $M \otimes \mathcal{W}_D \to M$) and that in the second sense (i.e., as the addition of tangent vectors to the space $[M \otimes \mathcal{W}_{D^p} \to M]$ at δ^p_M) coincide.*

Proof. This follows mainly from the following exponential law:
$$[M \otimes \mathcal{W}_{D^p} \to M \otimes \mathcal{W}_{D(2)}]$$
$$= [M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_{D(2)}$$

The details can safely be left to the reader. ∎

Unless stated to the contrary, we will use the terms *tangent-vector-valued p-semiforms on M* and *tangent-vector-valued p-forms on M* in the second sense.

4 The Frölicher-Nijenhuis Bracket

4.1 The Jacobi Identity for the Lie Bracket

Let us begin this subsection with the following definition.

Definition 31 *Given $\eta_1 \in [M \otimes \mathcal{W}_{D^p} \to M]$ and $\eta_2 \in [M \otimes \mathcal{W}_{D^q} \to M]$, two kinds of convolution, both of which belong to $[M \otimes \mathcal{W}_{D^{p+q}} \to M]$, are defined. The first, to be denoted by $\eta_1 * \eta_2$, is defined to be*

$$M \otimes \mathcal{W}_{D^{p+q}} = M \otimes (\mathcal{W}_{D^p} \otimes_\infty \mathcal{W}_{D^q}) = M \otimes (\mathcal{W}_{D^q} \otimes_\infty \mathcal{W}_{D^p})$$
$$= (M \otimes \mathcal{W}_{D^q}) \otimes \mathcal{W}_{D^p} \overset{\eta_2 \otimes \mathrm{id}_{\mathcal{W}_{D^p}}}{\to} M \otimes \mathcal{W}_{D^p} \overset{\eta_1}{\to} M$$

The second, to be denoted by $\eta_1 \widetilde{*} \eta_2$, is defined to be

$$M \otimes \mathcal{W}_{D^{p+q}} = M \otimes (\mathcal{W}_{D^p} \otimes_\infty \mathcal{W}_{D^q})$$

$$= (M \otimes \mathcal{W}_{D^p}) \otimes \mathcal{W}_{D^q} \xrightarrow{\eta_1 \otimes \mathrm{id}_{\mathcal{W}_{D^q}}} M \otimes \mathcal{W}_{D^q} \xrightarrow{\eta_2} M$$

Remark 32   *1. Our two convolutions are reminiscent of the familiar ones in abstract harmonic analysis and the theory of Schwartz distributions.*

2. If $p = q = 0$, then

$$M \otimes \mathcal{W}_{D^p} = M \otimes \mathcal{W}_{D^q} = M \otimes \mathcal{W}_{D^{p+q}} = M$$

so that

$$[M \otimes \mathcal{W}_{D^p} \to M] = [M \otimes \mathcal{W}_{D^q} \to M] = [M \otimes \mathcal{W}_{D^{p+q}} \to M] = [M \to M]$$

in which we have

$$\eta_1 * \eta_2 = \eta_1 \circ \eta_2$$
$$\eta_1 \widetilde{*} \eta_2 = \eta_2 \circ \eta_1$$

Notation 33 *Given $\sigma \in \mathbb{S}_p$ and $\eta \in [M \otimes \mathcal{W}_{D^p} \to M]$, we let η^σ denote*

$$\eta \circ \left(\mathrm{id}_M \otimes \mathcal{W}_{(d_1,\ldots,d_p) \in D^p \mapsto (d_{\sigma(1)},\ldots,d_{\sigma(p)}) \in D^p} \right)$$

It should be obvious that

Proposition 34 *Given $\eta_1 \in [M \otimes \mathcal{W}_{D^p} \to M]$ and $\eta_2 \in [M \otimes \mathcal{W}_{D^q} \to M]$, we have*

$$(\eta_2 * \eta_1)^{\sigma_{p,q}} = \eta_1 \widetilde{*} \eta_2$$
$$(\eta_2 \widetilde{*} \eta_1)^{\sigma_{p,q}} = \eta_1 * \eta_2$$

where $\sigma_{p,q}$ is the permutation mapping the sequence $1,\ldots,q,q+1,\ldots,p+q$ to the sequence $q+1,\ldots,p+q,1,\ldots,q$, namely,

$$\sigma_{p,q} = \begin{pmatrix} 1 & \ldots & p & p+1 & \ldots & p+q \\ q+1 & \ldots & p+q & 1 & \ldots & q \end{pmatrix}$$

It should also be obvious that

Proposition 35 *Given $\eta_1 \in [M \otimes \mathcal{W}_{D^p} \to M]$, $\eta_2 \in [M \otimes \mathcal{W}_{D^q} \to M]$ and $\eta_3 \in [M \otimes \mathcal{W}_{D^r} \to M]$, we have*

$$(\eta_1 * \eta_2) * \eta_3 = \eta_1 * (\eta_2 * \eta_3)$$
$$(\eta_1 \widetilde{*} \eta_2) \widetilde{*} \eta_3 = \eta_1 \widetilde{*} (\eta_2 \widetilde{*} \eta_3)$$

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Remark 36 *This proposition enables us to write, e.g., $\eta_1 * \eta_2 * \eta_3$ without parentheses in place of $(\eta_1 * \eta_2) * \eta_3$ or $\eta_1 * (\eta_2 * \eta_3)$. Similarly for $\eta_1 \widetilde{*} \eta_2 \widetilde{*} \eta_3$.*

Definition 37 *The canonical projection $\pi_M^{M \otimes \mathcal{W}_{D^p}} : M \otimes \mathcal{W}_{D^p} \to M$ is called a (p-dimensional) Dirac distribution on M, which is to be denoted by δ_M^p*

The following proposition should be obvious.

Proposition 38 *If one of $\eta_1 \in [M \otimes \mathcal{W}_{D^p} \to M]$ and $\eta_2 \in [M \otimes \mathcal{W}_{D^q} \to M]$ is a Dirac distribution, then $\eta_1 * \eta_2$ and $\eta_1 \widetilde{*} \eta_2$ coincide. In particular, if both of η_1 and η_2 are Dirac distributions, then $\eta_1 * \eta_2 = \eta_1 \widetilde{*} \eta_2$ is also a Dirac distribution.*

Definition 39 *An element $\xi \in [M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_{D^n}$ with*

$$\pi_{[M \otimes \mathcal{W}_{D^p} \to M]}^{[M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_{D^n}} (\xi) = \delta_M^p$$

is called an (n,p)-icon on M.

Remark 40 *By dropping the second and third conditions in the second definition of a tangent-vector-valued p-form on M in the preceding section, we rediscover the notion of a $(1,p)$-icon on M.*

Definition 41 *We define a binary mapping*

$$\circledast : ([M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_{D^m}) \times ([M \otimes \mathcal{W}_{D^q} \to M] \otimes \mathcal{W}_{D^n}) \to$$
$$[M \otimes \mathcal{W}_{D^{p+q}} \to M] \otimes \mathcal{W}_{D^{m+n}}$$

to be

$$([M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_{D^m}) \times ([M \otimes \mathcal{W}_{D^q} \to M] \otimes \mathcal{W}_{D^n})$$
$$\xrightarrow{\begin{array}{c} \left(\mathrm{id}_{[M \otimes \mathcal{W}_{D^p} \to M]} \otimes \mathcal{W}_{(d_1,\ldots,d_m,d_{m+1},\ldots,d_{m+n}) \in D^{m+n} \mapsto (d_1,\ldots,d_m) \in D^m} \right) \times \\ \left(\mathrm{id}_{[M \otimes \mathcal{W}_{D^q} \to M]} \otimes \mathcal{W}_{(d_1,\ldots,d_m,d_{m+1},\ldots,d_{m+n}) \in D^{m+n} \mapsto (d_{m+1},\ldots,d_{m+n}) \in D^n} \right) \end{array}}$$
$$([M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_{D^{m+n}}) \times ([M \otimes \mathcal{W}_{D^q} \to M] \otimes \mathcal{W}_{D^{m+n}})$$
$$= ([M \otimes \mathcal{W}_{D^p} \to M] \times [M \otimes \mathcal{W}_{D^q} \to M]) \otimes \mathcal{W}_{D^{m+n}}$$
$$\xrightarrow{* \otimes \mathrm{id}_{\mathcal{W}_{D^{m+n}}}} [M \otimes \mathcal{W}_{D^{p+q}} \to M] \otimes \mathcal{W}_{D^{m+n}}$$

Definition 42 *We define a binary mapping*

$$\widetilde{\circledast} : ([M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_{D^m}) \times ([M \otimes \mathcal{W}_{D^q} \to M] \otimes \mathcal{W}_{D^n}) \to$$
$$[M \otimes \mathcal{W}_{D^{p+q}} \to M] \otimes \mathcal{W}_{D^{m+n}}$$

to be

$$([M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_{D^m}) \times ([M \otimes \mathcal{W}_{D^q} \to M] \otimes \mathcal{W}_{D^n})$$

$$\xrightarrow{\begin{array}{c}\left(\mathrm{id}_{[M \otimes \mathcal{W}_{D^p} \to M]} \otimes \mathcal{W}_{(d_1,\ldots,d_m,d_{m+1},\ldots,d_{m+n}) \in D^{m+n} \mapsto (d_1,\ldots,d_m) \in D^m}\right) \times \\ \left(\mathrm{id}_{[M \otimes \mathcal{W}_{D^q} \to M]} \otimes \mathcal{W}_{(d_1,\ldots,d_m,d_{m+1},\ldots,d_{m+n}) \in D^{m+n} \mapsto (d_{m+1},\ldots,d_{m+n}) \in D^n}\right)\end{array}}$$

$$([M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_{D^{m+n}}) \times ([M \otimes \mathcal{W}_{D^q} \to M] \otimes \mathcal{W}_{D^{m+n}})$$

$$= ([M \otimes \mathcal{W}_{D^p} \to M] \times [M \otimes \mathcal{W}_{D^q} \to M]) \otimes \mathcal{W}_{D^{m+n}}$$

$$\xrightarrow{\tilde{\circledast} \otimes \mathrm{id}_{\mathcal{W}_{D^{m+n}}}} [M \otimes \mathcal{W}_{D^{p+q}} \to M] \otimes \mathcal{W}_{D^{m+n}}$$

Proposition 43 *Given* $\xi_1 \in [M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_{D^l}$, $\xi_2 \in [M \otimes \mathcal{W}_{D^q} \to M] \otimes \mathcal{W}_{D^m}$ *and* $\xi_3 \in [M \otimes \mathcal{W}_{D^r} \to M] \otimes \mathcal{W}_{D^n}$, *we have*

$$(\xi_1 \circledast \xi_2) \circledast \xi_3 = \xi_1 \circledast (\xi_2 \circledast \xi_3)$$
$$(\xi_1 \tilde{\circledast} \xi_2) \tilde{\circledast} \xi_3 = \xi_1 \tilde{\circledast} (\xi_2 \tilde{\circledast} \xi_3)$$

It should be obvious that

Lemma 44 *For any* $(1,p)$*-icon* ξ_1 *on* M *and any* $(1,q)$*-icon* ξ_2 *on* M, *we have*

$$\left(\mathrm{id}_{[M \otimes \mathcal{W}_{D^p} \to M]} \otimes \mathcal{W}_{(d_1,d_2) \in D(2) \mapsto (d_1,d_2) \in D^2}\right) (\xi_1 \circledast \xi_2)$$
$$= \left(\mathrm{id}_{[M \otimes \mathcal{W}_{D^p} \to M]} \otimes \mathcal{W}_{(d_1,d_2) \in D(2) \mapsto (d_1,d_2) \in D^2}\right) (\xi_1 \tilde{\circledast} \xi_2)$$

Therefore the following definition is meaningful.

Definition 45 *For any* $(1,p)$*-icon* ξ_1 *on* M *and any* $(1,q)$*-icon* ξ_2 *on* M, *their Lie bracket* $[\xi_1, \xi_2]_L \in [M \otimes \mathcal{W}_{D^{p+q}} \to M] \otimes \mathcal{W}_D$ *is defined to be*

$$[\xi_1, \xi_2]_L = \xi_1 \tilde{\circledast} \xi_2 \dot{-} \xi_1 \circledast \xi_2$$

It is easy to see that

Lemma 46 *In the above definition,* $[\xi_1, \xi_2]_L$ *is always a* $(1, p+q)$*-icon on* M.

Proposition 47 *If* ξ_1 *is a tangent-vector-valued* p*-semiform on* M *and* ξ_2 *is a tangent-vector-valued* q*-semiform on* M, *then we have*

$$\left(\left(\left(\alpha \cdot_i\right)_{M \otimes \mathcal{W}_{D^{p+q}}}\right)^* \otimes \mathrm{id}_{\mathcal{W}_{D^2}}\right) (\xi_1 \circledast \xi_2)$$
$$= \left(\mathrm{id}_{[M \otimes \mathcal{W}_{D^{p+q}} \to M]} \otimes \mathcal{W}_{(d_1,d_2) \in D^2 \mapsto (\alpha d_1, d_2) \in D^2}\right) (\xi_1 \circledast \xi_2)$$
$$\left(\left(\left(\alpha \cdot_i\right)_{M \otimes \mathcal{W}_{D^{p+q}}}\right)^* \otimes \mathrm{id}_{\mathcal{W}_{D^2}}\right) (\xi_1 \tilde{\circledast} \xi_2)$$
$$= \left(\mathrm{id}_{[M \otimes \mathcal{W}_{D^{p+q}} \to M]} \otimes \mathcal{W}_{(d_1,d_2) \in D^2 \mapsto (\alpha d_1, d_2) \in D^2}\right) (\xi_1 \tilde{\circledast} \xi_2)$$

for any natural number i with $1 \leq i \leq p$, while we have

$$\left(\left(\left(\alpha \cdot\right)_i\right)_{M \otimes \mathcal{W}_{D^{p+q}}}\right)^* \otimes \mathrm{id}_{\mathcal{W}_{D^2}}\right)(\xi_1 \circledast \xi_2)$$
$$= \left(\mathrm{id}_{[M \otimes \mathcal{W}_{D^{p+q}} \to M]} \otimes \mathcal{W}_{(d_1,d_2) \in D^2 \mapsto (d_1, \alpha d_2) \in D^2}\right)(\xi_1 \circledast \xi_2)$$
$$\left(\left(\left(\alpha \cdot\right)_i\right)_{M \otimes \mathcal{W}_{D^{p+q}}}\right)^* \otimes \mathrm{id}_{\mathcal{W}_{D^2}}\right)(\xi_1 \widetilde{\circledast} \xi_2)$$
$$= \left(\mathrm{id}_{[M \otimes \mathcal{W}_{D^{p+q}} \to M]} \otimes \mathcal{W}_{(d_1,d_2) \in D^2 \mapsto (d_1, \alpha d_2) \in D^2}\right)(\xi_1 \widetilde{\circledast} \xi_2)$$

for any natural number i with $p + 1 \leq i \leq p + q$.

Corollary 48 *If ξ_1 is a tangent-vector-valued p-semiform on M and ξ_2 is a tangent-vector-valued q-semiform on M, then $[\xi_1, \xi_2]_L$ is a tangent-vector-valued $(p+q)$-semiform on M.*

Proof. It suffices to see that

$$\left(\left(\left(\alpha \cdot\right)_i\right)_{M \otimes \mathcal{W}_{D^{p+q}}}\right)^* \otimes \mathrm{id}_{\mathcal{W}_D}\right)([\xi_1, \xi_2]_L)$$
$$= \left(\mathrm{id}_{[M \otimes \mathcal{W}_{D^{p+q}} \to M]} \otimes \mathcal{W}_{d \in D \mapsto \alpha d \in D}\right)([\xi_1, \xi_2]_L)$$

for any $\alpha \in \mathbb{R}$ and any natural number i with $1 \leq i \leq p+q$, which follows easily from the above Proposition and Proposition 5 in §3.4 of Lavendhomme [9]. ∎

Proposition 49 *If ξ_1, ξ_1' are tangent-vector-valued p-semiforms on M and ξ_2, ξ_2' are tangent-vector-valued q-semiforms on M with $\alpha \in \mathbb{R}$, then we have the following:*

1.
$$[\alpha \xi_1, \xi_2]_L = \alpha [\xi_1, \xi_2]_L$$

2.
$$[\xi_1 + \xi_1', \xi_2]_L = [\xi_1, \xi_2]_L + [\xi_1', \xi_2]_L$$

3.
$$[\xi_1, \alpha \xi_2]_L = \alpha [\xi_1, \xi_2]_L$$

4.
$$[\xi_1, \xi_2 + \xi_2']_L = [\xi_1, \xi_2]_L + [\xi_1, \xi_2']_L$$

Proof. The statements 1 and 3 follow from Proposition 5 in §3.4 of Lavendhomme [9], while the statements 2 and 4 follow from the statements 1 and 3 respectively. ∎

Notation 50 *Given* $\xi \in [M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_{D^n}$ *and* $\sigma \in \mathbb{S}_p$, ξ^σ *denotes*

$$\left(()^\sigma_{[M \otimes \mathcal{W}_{D^p} \to M]} \otimes \mathrm{id}_{\mathcal{W}_{D^n}} \right) (\xi)$$

where $()^\sigma_{[M \otimes \mathcal{W}_{D^p} \to M]} : [M \otimes \mathcal{W}_{D^p} \to M] \to [M \otimes \mathcal{W}_{D^p} \to M]$ *denotes the operation*

$$\eta \in [M \otimes \mathcal{W}_{D^p} \to M] \mapsto \eta \circ \left(\mathrm{id}_M \otimes \mathcal{W}_{(d_1,\ldots,d_p) \in D^p \mapsto (d_{\sigma(1)},\ldots,d_{\sigma(p)}) \in D^p} \right)$$

We will show that the Lie bracket $[\,]_L$ is antisymmetric.

Proposition 51 *Let* ξ_1 *be a* $(1,p)$*-icon on* M *and* ξ_2 *a* $(1,q)$*-icon on* M. *Then we have the following antisymmetry:*

$$[\xi_1, \xi_2]_L + ([\xi_2, \xi_1]_L)^{\sigma_{p,q}} = 0$$

Proof. This follows from Propositions 4 and 6 in §3.4 of Lavendhomme [9]. More specifically we have

$$[\xi_1, \xi_2]_L + ([\xi_2, \xi_1]_L)^{\sigma_{p,q}}$$
$$= (\xi_1 \widetilde{\circledast} \xi_2 \overset{\cdot}{-} \xi_1 \circledast \xi_2) + \left((\xi_2 \widetilde{\circledast} \xi_1)^{\sigma_{p,q}} \overset{\cdot}{-} (\xi_2 \circledast \xi_1)^{\sigma_{p,q}} \right)$$
$$= (\xi_1 \widetilde{\circledast} \xi_2 \overset{\cdot}{-} \xi_1 \circledast \xi_2) + \left(\xi_1 \circledast \xi_2 \overset{\cdot}{-} \xi_1 \widetilde{\circledast} \xi_2 \right)$$

[By Proposition 34]
$$= 0$$

∎

Theorem 52 *Let* ξ_1 *be a* $(1,p)$*-icon on* M, ξ_2 *a* $(1,q)$*-icon on* M, *and* ξ_3 *a* $(1,r)$*-icon on* M. *Then we have the following Jacobi identity:*

$$[\xi_1, [\xi_2, \xi_3]_L]_L + \left([\xi_2, [\xi_3, \xi_1]_L]_L \right)^{\sigma_{p,q+r}} + \left([\xi_3, [\xi_1, \xi_2]_L]_L \right)^{\sigma_{r,p+q}} = 0$$

In order to establish the above theorem, we need the following simple lemma, which is a tiny generalization of Proposition 2.6 of [13].

Lemma 53 *Let* ξ *be an* $(1,p)$*-icon on* M, *and* ξ_1 *and* ξ_2 $(2,q)$*-icons on* M *with*

$$\left(\mathrm{id}_{[M \otimes \mathcal{W}_{D^q} \to M]} \otimes \mathcal{W}_{(d_1,d_2) \in D(2) \mapsto (d_1,d_2) \in D^2} \right) (\xi_1)$$
$$= \left(\mathrm{id}_{[M \otimes \mathcal{W}_{D^q} \to M]} \otimes \mathcal{W}_{(d_1,d_2) \in D(2) \mapsto (d_1,d_2) \in D^2} \right) (\xi_2)$$

Then the following formulas are both meaningful and valid.

$$\xi \circledast \xi_1 \overset{\cdot}{\underset{1}{-}} \xi \circledast \xi_2 = \xi \circledast (\xi_1 \overset{\cdot}{-} \xi_2)$$

$$\xi \widetilde{\circledast} \xi_1 \overset{\cdot}{\underset{1}{-}} \xi \widetilde{\circledast} \xi_2 = \xi \widetilde{\circledast} (\xi_1 \overset{\cdot}{-} \xi_2)$$

$$\xi_1 \circledast \xi \overset{\cdot}{\underset{3}{-}} \xi_2 \circledast \xi = (\xi_1 \overset{\cdot}{-} \xi_2) \circledast \xi$$

$$\xi_1 \widetilde{\circledast} \xi \overset{\cdot}{\underset{3}{-}} \xi_2 \widetilde{\circledast} \xi = (\xi_1 \overset{\cdot}{-} \xi_2) \widetilde{\circledast} \xi$$

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Proof. (of Theorem 52). Our present discussion is a tiny generalization of Proposition 2.7 in [13]. We define six $(3, p+q+r)$-icons on M as follows:

$$\xi_{123} = \xi_1 \circledast \xi_2 \circledast \xi_3$$
$$\xi_{132} = \xi_1 \circledast (\xi_2 \widetilde{\circledast} \xi_3)$$
$$\xi_{213} = (\xi_1 \widetilde{\circledast} \xi_2) \circledast \xi_3$$
$$\xi_{231} = \xi_1 \widetilde{\circledast} (\xi_2 \circledast \xi_3)$$
$$\xi_{312} = (\xi_1 \circledast \xi_2) \widetilde{\circledast} \xi_3$$
$$\xi_{321} = \xi_1 \widetilde{\circledast} \xi_2 \widetilde{\circledast} \xi_3$$

Then it is easy, by dint of Lemma 53, to see that

$$[\xi_1, [\xi_2, \xi_3]_L]_L = (\xi_{123} \underset{1}{\overset{\cdot}{-}} \xi_{132}) - (\xi_{231} \underset{1}{\overset{\cdot}{-}} \xi_{321}) \tag{5}$$

$$\left([\xi_2, [\xi_3, \xi_1]_L]_L\right)^{\sigma_{p,q+r}} = (\xi_{231} \underset{2}{\overset{\cdot}{-}} \xi_{213}) - (\xi_{312} \underset{2}{\overset{\cdot}{-}} \xi_{132}) \tag{6}$$

$$\left([\xi_3, [\xi_1, \xi_2]_L]_L\right)^{\sigma_{r,p+q}} = (\xi_{312} \underset{3}{\overset{\cdot}{-}} \xi_{321}) - (\xi_{123} \underset{3}{\overset{\cdot}{-}} \xi_{213}) \tag{7}$$

Therefore the desired Jacobi identity follows directly from the general Jacobi identity. ∎

Remark 54 *In order to see that the right-hand side of (5) is meaningful, we have to check that all of*

$$\xi_{123} \underset{1}{\overset{\cdot}{-}} \xi_{132}$$

$$\xi_{231} \underset{1}{\overset{\cdot}{-}} \xi_{321}$$

$$(\xi_{123} \underset{1}{\overset{\cdot}{-}} \xi_{132}) \overset{\cdot}{-} (\xi_{231} \underset{1}{\overset{\cdot}{-}} \xi_{321})$$

are meaningful. Since $\xi_2 \circledast \xi_3 \overset{\cdot}{-} \xi_2 \widetilde{\circledast} \xi_3$ is meaningful by Lemma 44, $\xi_{123} \underset{1}{\overset{\cdot}{-}} \xi_{132}$ is also meaningful and we have

$$\xi_{123} \underset{1}{\overset{\cdot}{-}} \xi_{132} = \xi_1 \circledast (\xi_2 \circledast \xi_3 \overset{\cdot}{-} \xi_2 \widetilde{\circledast} \xi_3)$$

by Lemma 53. Similarly $\xi_{231} \underset{1}{\overset{\cdot}{-}} \xi_{321}$ is meaningful and we have

$$\xi_{231} \underset{1}{\overset{\cdot}{-}} \xi_{321} = \xi_1 \widetilde{\circledast} (\xi_2 \circledast \xi_3 \overset{\cdot}{-} \xi_2 \widetilde{\circledast} \xi_3)$$

Therefore $(\xi_{123} \underset{1}{\overset{\cdot}{-}} \xi_{132}) - (\xi_{231} \underset{1}{\overset{\cdot}{-}} \xi_{321})$ is meaningful by Lemma 44. Similar considerations apply to (6) and (7).

4.2 The Jacobi Identity for the Frölicher-Nijenhuis Bracket

Definition 55 *Given a $(1,p)$-icon ξ on M, we define another $(1,p)$-icon $\mathcal{A}\xi$ on M to be*

$$\mathcal{A}\xi = \sum_{\sigma \in \mathbb{S}_p} \varepsilon_\sigma \xi^\sigma$$

Notation 56 *Given a $(1,p+q)$-icon ξ on M, we write $\mathcal{A}_{p,q}\xi$ for $(1/p!q!)\mathcal{A}\xi$. Given a $(1,p+q+r)$-icon ξ on M, We write $\mathcal{A}_{p,q,r}\xi$ for $(1/p!q!r!)\mathcal{A}\xi$.*

Lemma 57 *If ξ_1 is a tangent-vector-valued p-form on M, ξ_2 is a tangent-vector-valued q-form on M and ξ_3 is a tangent-vector-valued r-form on M, then we have*

$$\mathcal{A}_{p,q+r}([\xi_1, \mathcal{A}_{q,r}([\xi_2,\xi_3]_L)]_L) = \mathcal{A}_{p,q,r}([\xi_1, [\xi_2,\xi_3]_L]_L)$$

Proof. By the same token as in establishing the familiar associativity of wedge products in differential forms. ■

Definition 58 *Given a tangent-vector-valued p-form ξ_1 on M and a tangent-vector-valued q-form ξ_2 on M, we are going to define their Frölicher-Nijenhuis bracket $[\xi_1,\xi_2]_{FN}$ to be*

$$[\xi_1,\xi_2]_{FN} = \mathcal{A}_{p,q}([\xi_1,\xi_2]_L)$$

which is undoubtedly a tangent-vector-valued $(p+q)$-form on M.

Proposition 59 *If ξ_1 is a tangent-vector-valued p-form on M and ξ_2 is a tangent-vector-valued q-form on M, then we have the following graded antisymmetry:*

$$[\xi_1,\xi_2]_{FN} = -(-1)^{pq}[\xi_2,\xi_1]_{FN}$$

Proof. We have

$$
\begin{aligned}
&[\xi_1,\xi_2]_{FN}\\
&= \mathcal{A}_{p,q}([\xi_1,\xi_2]_L)\\
&= -\mathcal{A}_{p,q}(([\xi_2,\xi_1]_L)^{\sigma_{p,q}}) \quad \text{[By Proposition 51]}\\
&= -\frac{1}{p!q!} \sum_{\tau \in \mathbb{S}_{p+q}} \varepsilon_\tau (([\xi_2,\xi_1]_L)^{\sigma_{p,q}})^\tau\\
&= -\frac{1}{p!q!} \sum_{\tau \in \mathbb{S}_{p+q}} \varepsilon_\tau ([\xi_2,\xi_1]_L)^{\tau \sigma_{p,q}}\\
&= -\frac{1}{p!q!} \varepsilon_{\sigma_{p,q}} \sum_{\tau \in \mathbb{S}_{p+q}} \varepsilon_{\tau \sigma_{p,q}} ([\xi_2,\xi_1]_L)^{\tau \sigma_{p,q}}\\
&= -\varepsilon_{\sigma_{p,q}} [\xi_2,\xi_1]_{FN}
\end{aligned}
$$

Since $\varepsilon_\rho = (-1)^{pq}$, the desired conclusion follows. ■

Theorem 60 *If ξ_1 is a tangent-vector-valued p-form on M, ξ_2 is a tangent-vector-valued q-form on M and ξ_3 is a tangent-vector-valued r-form on M, then the following graded Jacobi identity holds:*

$$[\xi_1, [\xi_2, \xi_3]_{FN}]_{FN} + (-1)^{p(q+r)} [\xi_2, [\xi_3, \xi_1]_{FN}]_{FN} + (-1)^{r(p+q)} [\xi_3, [\xi_1, \xi_2]_{FN}]_{FN} = 0$$

Proof. We have

$[\xi_1, [\xi_2, \xi_3]_{FL}]_{FL} + (-1)^{p(q+r)} [\xi_2, [\xi_3, \xi_1]_{FL}]_{FL} + (-1)^{r(p+q)} [\xi_3, [\xi_1, \xi_2]_{FL}]_{FL}$

$= \mathcal{A}_{p,q+r}([\xi_1, \mathcal{A}_{q,r}([\xi_2, \xi_3]_L)]_L) + (-1)^{p(q+r)} \mathcal{A}_{q,p+r}([\xi_2, \mathcal{A}_{p,r}([\xi_3, \xi_1]_L)]_L) +$

$(-1)^{r(p+q)} \mathcal{A}_{r,p+q}([\xi_3, \mathcal{A}_{p,q}([\xi_1, \xi_2]_L)]_L)$

$= \mathcal{A}_{p,q,r} \left\{ [\xi_1, [\xi_2, \xi_3]_L]_L + (-1)^{p(q+r)} [\xi_2, [\xi_3, \xi_1]_L]_L + (-1)^{r(p+q)} [\xi_3, [\xi_1, \xi_2]_L]_L \right\}$

[By Lemma 57]

$= \mathcal{A}_{p,q,r} \left\{ [\xi_1, [\xi_2, \xi_3]_L]_L + ([\xi_2, [\xi_3, \xi_1]_L]_L)^{\sigma_{p,q+r}} + ([\xi_3, [\xi_1, \xi_2]_L]_L)^{\sigma_{r,p+q}} \right\}$

$= 0$

[By Theorem 52]

∎

5   The Lie Derivation

5.1   The Lie Derivation of the First Type

Definition 61 *Given $\eta \in [M \otimes \mathcal{W}_{D^p} \to M]$ and $\theta \in [M \otimes \mathcal{W}_{D^q} \to \mathbb{R}]$, their convolution $\eta \widetilde{*} \theta \in [M \otimes \mathcal{W}_{D^{p+q}} \to \mathbb{R}]$ is defined to be the outcome of the composition of mappings*

$$M \otimes \mathcal{W}_{D^{p+q}} = M \otimes (\mathcal{W}_{D^p} \otimes_\infty \mathcal{W}_{D^q})$$

$$= (M \otimes \mathcal{W}_{D^p}) \otimes \mathcal{W}_{D^q} \overset{\eta \otimes \mathrm{id}_{\mathcal{W}_{D^q}}}{\to} M \otimes \mathcal{W}_{D^q} \overset{\theta}{\to} \mathbb{R}$$

It should be obvious that

Proposition 62 *Given $\eta_1 \in [M \otimes \mathcal{W}_{D^p} \to M]$, $\eta_2 \in [M \otimes \mathcal{W}_{D^q} \to M]$ and $\theta \in [M \otimes \mathcal{W}_{D^r} \to \mathbb{R}]$, we have*

$$(\eta_1 \widetilde{*} \eta_2) \widetilde{*} \theta = \eta_1 \widetilde{*} (\eta_2 \widetilde{*} \theta)$$

Definition 63 *We define a binary mapping*

$$\widetilde{\circledast} : ([M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_{D^m}) \times ([M \otimes \mathcal{W}_{D^q} \to \mathbb{R}] \otimes \mathcal{W}_{D^n}) \to$$
$$[M \otimes \mathcal{W}_{D^{p+q}} \to \mathbb{R}] \otimes \mathcal{W}_{D^{m+n}}$$

to be the composition of mappings

$$([M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_{D^m}) \times ([M \otimes \mathcal{W}_{D^q} \to \mathbb{R}] \otimes \mathcal{W}_{D^n})$$

$$\xrightarrow{\left(\mathrm{id}_{[M \otimes \mathcal{W}_{D^p} \to M]} \otimes \mathcal{W}_{(d_1,\ldots,d_m,d_{m+1},\ldots,d_{m+n}) \in D^{m+n} \mapsto (d_1,\ldots,d_m) \in D^m}\right) \times}$$
$$\overrightarrow{\left(\mathrm{id}_{[M \otimes \mathcal{W}_{D^q} \to \mathbb{R}]} \otimes \mathcal{W}_{(d_1,\ldots,d_m,d_{m+1},\ldots,d_{m+n}) \in D^{m+n} \mapsto (d_{m+1},\ldots,d_{m+n}) \in D^n}\right)}$$

$$([M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_{D^{m+n}}) \times ([M \otimes \mathcal{W}_{D^q} \to \mathbb{R}] \otimes \mathcal{W}_{D^{m+n}})$$
$$= ([M \otimes \mathcal{W}_{D^p} \to M] \times [M \otimes \mathcal{W}_{D^q} \to \mathbb{R}]) \otimes \mathcal{W}_{D^{m+n}}$$
$$\xrightarrow{\widetilde{\divideontimes} \otimes \mathrm{id}_{\mathcal{W}_{D^{m+n}}}} [M \otimes \mathcal{W}_{D^{p+q}} \to \mathbb{R}] \otimes \mathcal{W}_{D^{m+n}}$$

It should be obvious that

Proposition 64 *Given* $\xi_1 \in [M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_{D^m}$, $\xi_2 \in [M \otimes \mathcal{W}_{D^q} \to M] \otimes \mathcal{W}_{D^n}$ *and* $\theta \in [M \otimes \mathcal{W}_{D^r} \to \mathbb{R}] \otimes \mathcal{W}_{D^l}$, *we have*

$$(\xi_1 \widetilde{\divideontimes} \xi_2) \widetilde{\divideontimes} \theta = \xi_1 \widetilde{\divideontimes} (\xi_2 \widetilde{\divideontimes} \theta)$$

Definition 65 *For any* $\xi \in [M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_D$ *and any* $\theta \in [M \otimes \mathcal{W}_{D^q} \to \mathbb{R}]$, *we define* $\widehat{\mathcal{L}}_\xi \theta$ *to be*

$$\widehat{\mathcal{L}}_\xi \theta = \mathbf{D}\left(\xi \widetilde{\otimes} \theta\right)$$

It is easy to see that

Proposition 66 *Given* $\xi \in [M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_D$, $\theta_1 \in [M \otimes \mathcal{W}_{D^q} \to \mathbb{R}]$ *and* $\theta_2 \in [M \otimes \mathcal{W}_{D^r} \to \mathbb{R}]$, *we have*

$$\widehat{\mathcal{L}}_\xi (\theta_1 \otimes \theta_2) = \left(\widehat{\mathcal{L}}_\xi \theta_1\right) \otimes \theta_2 + \left(\theta_1 \otimes \left(\widehat{\mathcal{L}}_\xi \theta_2\right)\right)^{\sigma_{p,q}^{+r}}$$

where $\sigma_{p,q}^{+r}$ *is*

$$\begin{pmatrix} 1 & \ldots & p & p+1 & \ldots & p+q & p+q+1 & \ldots & p+q+r \\ q+1 & \ldots & p+q & 1 & \ldots & q & p+q+1 & \ldots & p+q+r \end{pmatrix}$$

It should be obvious that

Proposition 67 *If* $\xi \in [M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_D$ *is a tangent-vector-valued* p-*semiform and* $\theta \in [M \otimes \mathcal{W}_{D^q} \to \mathbb{R}]$ *is a* q-*semiform, then* $\widehat{\mathcal{L}}_\xi \theta$ *is a* $(p+q)$-*semiform.*

Remark 68 *Therefore, given a tangent-vector-valued* p-*semiform* ξ *on* M, $\widehat{\mathcal{L}}_\xi$ *is considered to be a graded mapping of degree* p *on the space* $\widetilde{\Omega}(M)$.

Proposition 69 *If* $\xi, \xi_1, \xi_2 \in [M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_D$ *are tangent-vector-valued* p-*semiforms,* $\theta, \theta_1, \theta_2 \in [M \otimes \mathcal{W}_{D^q} \to \mathbb{R}]$ *are* q-*semiforms and* $\alpha \in \mathbb{R}$, *then we have the following:*

1.

$$\widehat{\mathcal{L}}_{\xi_1+\xi_2}\theta = \widehat{\mathcal{L}}_{\xi_1}\theta + \widehat{\mathcal{L}}_{\xi_2}\theta$$

2.

$$\widehat{\mathcal{L}}_{\alpha\xi}\theta = \alpha\left(\widehat{\mathcal{L}}_{\xi}\theta\right)$$

3.

$$\widehat{\mathcal{L}}_{\xi}\left(\theta_1 + \theta_2\right) = \widehat{\mathcal{L}}_{\xi}\theta_1 + \widehat{\mathcal{L}}_{\xi}\theta_2$$

4.

$$\widehat{\mathcal{L}}_{\xi}\left(\alpha\theta\right) = \alpha\left(\widehat{\mathcal{L}}_{\xi}\theta\right)$$

Proof. The statements 2 and 4 follow from the definitions. The statement 1 follows from the statement 2, while the statement 3 follows from the statement 4. ∎

Theorem 70 *If $\xi_1 \in [M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_D$ is a tangent-vector-valued p-semiform and $\xi_2 \in [M \otimes \mathcal{W}_{D^q} \to M] \otimes \mathcal{W}_D$ is a tangent-vector-valued q-semiform, then we have*

$$\widehat{\mathcal{L}}_{[\xi_1,\xi_2]_L} = \left[\widehat{\mathcal{L}}_{\xi_1}, \widehat{\mathcal{L}}_{\xi_2}\right] = \widehat{\mathcal{L}}_{\xi_1} \circ \widehat{\mathcal{L}}_{\xi_2} - (-1)^{pq}\widehat{\mathcal{L}}_{\xi_2} \circ \widehat{\mathcal{L}}_{\xi_1}$$

Proof. If $\theta \in [M \otimes \mathcal{W}_{D^r} \to \mathbb{R}]$ is a r-semiform, then we have

$$\widehat{\mathcal{L}}_{[\xi_1,\xi_2]_L}\theta$$
$$= \mathbf{D}\left([\xi_1,\xi_2]_L \,\widetilde{\otimes}\theta\right)$$
$$= \mathbf{D}\left(\left(\xi_1\widetilde{\otimes}\xi_2 \stackrel{.}{-} \xi_1 \otimes \xi_2\right)\widetilde{\circledast}\theta\right)$$
$$= \mathbf{D}\left(\left(\xi_1\widetilde{\otimes}\xi_2\right)\widetilde{\otimes}\theta \stackrel{.}{-} \left(\xi_1 \otimes \xi_2\right)\widetilde{\otimes}\theta\right)$$
$$= \mathbf{D}\left(\left(\xi_1\widetilde{\otimes}\xi_2\right)\widetilde{\otimes}\theta \stackrel{.}{-} \left(\xi_2\widetilde{\otimes}\xi_1\right)_{\tau}^{\sigma_{p,q}}\widetilde{\otimes}\theta\right)$$
$$= \mathbf{D}\left(\begin{array}{c}\xi_1\widetilde{\otimes}\left(\xi_2\widetilde{\otimes}\theta\right)\stackrel{.}{-}\\\left(\mathrm{id}_{\widetilde{\Omega}^{p+q+r}(M)}\otimes\mathcal{W}_{(d_1,d_2)\in D^2\mapsto(d_2,d_1)\in D^2}\right)\left(\left(\xi_2\widetilde{\otimes}\left(\xi_1\widetilde{\otimes}\theta\right)\right)^{\sigma_{p,q}^{+r}}\right)\end{array}\right)$$
$$= \mathbf{D}\left(\mathbf{D}_2\left(\xi_1\widetilde{\otimes}\left(\xi_2\widetilde{\otimes}\theta\right)\right)\right) - (-1)^{pq}\mathbf{D}\left(\mathbf{D}_1\left(\xi_2\widetilde{\otimes}\left(\xi_1\widetilde{\otimes}\theta\right)\right)\right)$$
[By Proposition 11]
$$= \mathbf{D}\left(\xi_1\widetilde{\otimes}\mathbf{D}\left(\xi_2\widetilde{\otimes}\theta\right)\right) - (-1)^{pq}\mathbf{D}\left(\xi_2\widetilde{\otimes}\mathbf{D}\left(\xi_1\widetilde{\otimes}\theta\right)\right)$$
$$= \widehat{\mathcal{L}}_{\xi_1}\left(\widehat{\mathcal{L}}_{\xi_2}\theta\right) - (-1)^{pq}\widehat{\mathcal{L}}_{\xi_2}\left(\widehat{\mathcal{L}}_{\xi_1}\theta\right)$$

∎

5.2   The Lie Derivation of the Second Type

Definition 71 *For any $\xi \in [M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_D$ and any $\theta \in [M \otimes \mathcal{W}_{D^q} \to \mathbb{R}]$, we define $\mathcal{L}_\xi \theta$ to be*

$$\mathcal{L}_\xi \theta = \mathcal{A}_{p,q}\left(\widehat{\mathcal{L}}_\xi \theta\right)$$

It should be obvious that

Proposition 72 *If $\xi \in [M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_D$ is a tangent-vector-valued p-form and $\theta \in [M \otimes \mathcal{W}_{D^q} \to \mathbb{R}]$ is a q-form, then $\mathcal{L}_\xi \theta$ is a $(p+q)$-form.*

Proposition 73 *Given $\xi \in [M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_D$, $\theta_1 \in [M \otimes \mathcal{W}_{D^q} \to \mathbb{R}]$ and $\theta_2 \in [M \otimes \mathcal{W}_{D^r} \to \mathbb{R}]$, we have the following:*

1.
$$\mathcal{A}_{p,q,r}\left(\left(\widehat{\mathcal{L}}_\xi \theta_1\right) \otimes \theta_2\right) = \mathcal{A}_{p+q,r}\left(\mathcal{A}_{p,q}\left(\widehat{\mathcal{L}}_\xi \theta_1\right) \otimes \theta_2\right)$$

2.
$$\mathcal{A}_{p,q,r}\left(\theta_1 \otimes \left(\widehat{\mathcal{L}}_\xi \theta_2\right)\right) = \mathcal{A}_{q,p+r}\left(\theta_1 \otimes \mathcal{A}_{p,r}\left(\widehat{\mathcal{L}}_\xi \theta_2\right)\right)$$

Proof. By the same token as that in establishing the familiar associativity of wedge products in differential forms. ∎

Proposition 74 *Given $\xi \in [M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_D$, $\theta_1 \in [M \otimes \mathcal{W}_{D^q} \to \mathbb{R}]$ and $\theta_2 \in [M \otimes \mathcal{W}_{D^r} \to \mathbb{R}]$, we have*

$$\mathcal{L}_\xi\left(\theta_1 \wedge \theta_2\right) = \left(\mathcal{L}_\xi \theta_1\right) \wedge \theta_2 + (-1)^{pq}\theta_1 \wedge \left(\mathcal{L}_\xi \theta_2\right)$$

Proof. We proceed as follows:

$$\mathcal{L}_\xi\left(\theta_1 \wedge \theta_2\right)$$
$$= \mathcal{A}_{p,q+r}\left(\widehat{\mathcal{L}}_\xi\left(\mathcal{A}_{q,r}\left(\theta_1 \otimes \theta_2\right)\right)\right)$$
$$= \mathcal{A}_{p,q,r}\left(\widehat{\mathcal{L}}_\xi\left(\theta_1 \otimes \theta_2\right)\right)$$
$$= \mathcal{A}_{p,q,r}\left(\left(\left(\widehat{\mathcal{L}}_\xi \theta_1\right) \otimes \theta_2 + \left(\theta_1 \otimes \left(\widehat{\mathcal{L}}_\xi \theta_2\right)\right)\right)^{\sigma_{p,q}^{+r}}\right)$$
[By Proposition 66]
$$= \mathcal{A}_{p,q,r}\left(\left(\widehat{\mathcal{L}}_\xi \theta_1\right) \otimes \theta_2\right) + \mathcal{A}_{p,q,r}\left(\left(\theta_1 \otimes \left(\widehat{\mathcal{L}}_\xi \theta_2\right)\right)^{\sigma_{p,q}^{+r}}\right)$$
$$= \mathcal{A}_{p+q,r}\left(\mathcal{A}_{p,q}\left(\widehat{\mathcal{L}}_\xi \theta_1\right) \otimes \theta_2\right) + (-1)^{pq}\mathcal{A}_{q,p+r}\left(\theta_1 \otimes \mathcal{A}_{p,r}\left(\widehat{\mathcal{L}}_\xi \theta_2\right)\right)$$
[By Proposition 73]
$$= \left(\mathcal{L}_\xi \theta_1\right) \wedge \theta_2 + (-1)^{pq}\theta_1 \wedge \left(\mathcal{L}_\xi \theta_2\right)$$

∎

Remark 75 *Therefore, given a tangent-vector-valued p-form ξ on M, $\mathcal{L}_\xi$ is considered to be a graded mapping of degree p on the space $\Omega(M)$.*

Proposition 76 *For any $\xi \in [M \otimes \mathcal{W}_{D^{p+q}} \to M] \otimes \mathcal{W}_D$, any $\xi_1 \in [M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_D$, any $\xi_2 \in [M \otimes \mathcal{W}_{D^q} \to M] \otimes \mathcal{W}_D$, and any $\theta \in [M \otimes \mathcal{W}_{D^r} \to \mathbb{R}]$, we have the following:*

1.
$$\mathcal{A}_{p,q,r}\left(\widehat{\mathcal{L}}_\xi \theta\right) = \mathcal{A}_{p+q,r}\left(\widehat{\mathcal{L}}_{\mathcal{A}_{p,q}(\xi)}\theta\right)$$

2.
$$\mathcal{A}_{p,q,r}\left(\widehat{\mathcal{L}}_{\xi_1}\left(\widehat{\mathcal{L}}_{\xi_2}\theta\right)\right) = \mathcal{A}_{p,q+r}\left(\widehat{\mathcal{L}}_{\xi_1}\mathcal{A}_{q,r}\left(\widehat{\mathcal{L}}_{\xi_2}\theta\right)\right)$$

3.
$$\mathcal{A}_{p,q,r}\left(\widehat{\mathcal{L}}_{\xi_2}\left(\widehat{\mathcal{L}}_{\xi_1}\theta\right)\right) = \mathcal{A}_{q,p+r}\left(\widehat{\mathcal{L}}_{\xi_2}\mathcal{A}_{p,r}\left(\widehat{\mathcal{L}}_{\xi_1}\theta\right)\right)$$

Proof. By the same token as that in the familiar associativity of wedge product in differential forms. ■

Theorem 77 *If both $\xi_1 \in [M \otimes \mathcal{W}_{D^p} \to M] \otimes \mathcal{W}_D$ and $\xi_2 \in [M \otimes \mathcal{W}_{D^q} \to M] \otimes \mathcal{W}_D$ are tangent-vector-valued semiforms, then we have*

$$\mathcal{L}_{[\xi_1,\xi_2]_{FN}} = [\mathcal{L}_{\xi_1}, \mathcal{L}_{\xi_2}] = \mathcal{L}_{\xi_1} \circ \mathcal{L}_{\xi_2} - (-1)^{pq}\mathcal{L}_{\xi_2} \circ \mathcal{L}_{\xi_1}$$

Proof. For any $\theta \in [M \otimes \mathcal{W}_{D^r} \to \mathbb{R}] \otimes \mathcal{W}_D$, we have

$\mathcal{L}_{[\xi_1,\xi_2]_{FN}}\theta$

$= \mathcal{A}_{p+q,r}\left(\widehat{\mathcal{L}}_{\mathcal{A}_{p,q}\left([\xi_1,\xi_2]_L\right)}\theta\right)$

$= \mathcal{A}_{p,q,r}\left(\widehat{\mathcal{L}}_{[\xi_1,\xi_2]_L}\theta\right)$

[By the first statement of Proposition 76]

$= \mathcal{A}_{p,q,r}\left(\widehat{\mathcal{L}}_{\xi_1}\left(\widehat{\mathcal{L}}_{\xi_2}\theta\right) - (-1)^{pq}\widehat{\mathcal{L}}_{\xi_2}\left(\widehat{\mathcal{L}}_{\xi_1}\theta\right)\right)$

[By Theorem 69]

$= \mathcal{A}_{p,q,r}\left(\widehat{\mathcal{L}}_{\xi_1}\left(\widehat{\mathcal{L}}_{\xi_2}\theta\right)\right) - (-1)^{pq}\mathcal{A}_{p,q,r}\left(\widehat{\mathcal{L}}_{\xi_2}\left(\widehat{\mathcal{L}}_{\xi_1}\theta\right)\right)$

$= \mathcal{A}_{p,q+r}\left(\widehat{\mathcal{L}}_{\xi_1}\mathcal{A}_{q,r}\left(\widehat{\mathcal{L}}_{\xi_2}\theta\right)\right) - (-1)^{pq}\mathcal{A}_{q,p+r}\left(\widehat{\mathcal{L}}_{\xi_2}\mathcal{A}_{p,r}\left(\widehat{\mathcal{L}}_{\xi_1}\theta\right)\right)$

[By the second and third statements of Proposition 76]

$= \mathcal{L}_{\xi_1}\left(\mathcal{L}_{\xi_2}\theta\right) - (-1)^{pq}\mathcal{L}_{\xi_2}\left(\mathcal{L}_{\xi_1}\theta\right)$

■

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